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Calculation of Metric for Gaussian Weave State

L. Shao · D. Shao · C.G. Shao · H. Noda

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Abstract Using the recoupling theorem and graph calculation in loop quantum gravity, it is demonstrated that the action of metric matrix operator on Gaussian weave state is an eigenaction, the representation matrix elements of the metric operator and their expectation values are calculated. The values of length of tangent vectors of edges adjacent to the vertex of Gaussian weave state, as well as the angles between them are also obtained in the cases of $k = 0$ and $k = 2$.

Keywords Gaussian weave · Metric matrix operator · Expectation value matrix of metric operator · Action of volume operator · Spin-geometry

As we know, it is a very important physical result in loop quantum gravity, that the spectra of geometrical operators, such as the operators of area of surface and of volume of region in space Σ, are purely discrete [1]. Weave states which are eigenstates of the geometrical operators, may be used to approximate smooth flat space at large scales [2]. At the kinematical level, the first attempt in this direction were given via a quantum operator $\hat{Q}$ in [3]. However, it was known that the operator $\hat{Q}$ does not capture more information than the area and volume operators. Some authors consider a basic building graph of a weave that the graph is made of two non-coplanar circles with same radius respect to a flat metric, which intersect in a single vertex. The basic graphs of circles are randomly distributed in finite region $R \subset \Sigma$ with average density ρ measured with the flat metric. Then, the weave can be fully determined, and the approximation scale may be calculated [4].

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Recently, Gaussian weave is studied in some works. Gaussian weave state in R is given by [5] $W=\prod_{\upsilon\in R}W_\upsilon$; for each vertex υ contributed from the basic graphs, the weave state is framed by

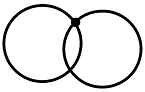
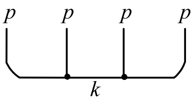
$$W_\upsilon=\sum_{p=0}^{\infty}C_p\psi_p=N\exp(-\lambda^2(\psi_1-2)^2),$$

where N,λ are normalization factor and Gaussian parameter, and ψ_p is known as basis of W_υ. In this paper, we shall give some calculations about Gaussian weave state. If we want to approximate a flat space with Gaussian weave, because each vertex $\upsilon\in R$ could be overlapped by infinite number of vertices of $\psi_p's$, the spin-geometry [6–8] of spin network shall give us useful help to determine the configuration of the weave state W_υ which may be used to investigate the weaving of space.

In the paper using the recoupling theorem and graph calculation, we shall prove that the action of metric operator on the Gauss weave state ψ_p is an eigenaction, and the representation matrix elements of the operator tare given, in Sects. 1 and 2. Sections 3–6 first introduce a generic recoupling matrix, then calculate the ones we shall use. The expectation values of volume operator and the metric operator are calculated in Sects. 7–9. In the final section, values of length of tangent vectors at the vertex υ of basis state ψ_p, and the angles between them are obtained.

1 The Eigenactions of Metric Matrix Diagonal Component Operators $\hat{M}(s_\alpha,s_\alpha)$ on Vertex ϕ_k

Denote the basis state ψ_p and its vertex υ as

 and $N_k(p)$ 

$=\phi_k(p,p,p,p)$,

where $N_k(p)=N_k(p,p,p,p)$ is the normalization factor of its associated vertex. Our spin-geometry calculations for Gaussian weave begin with the operations on the graph of vertex ϕ_k. The metric operator that acts on ϕ_k is given by

$$\hat{M}(s_\alpha,s_\beta)(\upsilon)=\frac{1}{2}[\hat{\Theta}^i_\alpha(\upsilon)\hat{\Theta}^i_\beta(\upsilon)+\alpha\leftrightarrow\beta],\tag{1}$$

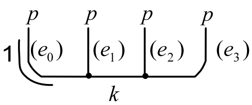
where

$$\hat{\Theta}^i_\alpha(\upsilon)=\frac{4}{i\hbar k}\operatorname{tr}(\tau^ih_\alpha\hat{V}h_\alpha^{-1}),\tag{2}$$

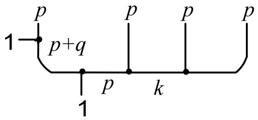
and holonomy $h_\alpha=h_{s_\alpha}$, S_α is a segment that endpoint is at vertex υ; τ^i is the generator of $su(2)$; $\hat{V}$ is the volume operator used in loop quantum gravity.

1.1 The Eigenaction of Operator $\hat{M}(s_0,s_0)$ on ϕ_k

First, we compute the action of holonomy h_0^{-1} on the 4-valent vertex $\phi_k(p,p,p,p)$, that is

$h_0^{-1}\phi_k(p,p,p,p)=N_k(p)$ 

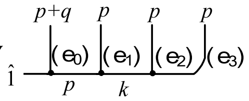
$$= N_k(P) \sum_{q=\pm 1} \gamma_q(P)$$



, (3)

where $\gamma_+(p) = 1$, $\gamma_-(p) = -\frac{P}{P+1}$. Define the action of the volume operator $\hat{V}$ on the last graph of expression (3) as

$$\hat{V} h_0^{-1} \phi_k = N_k(p) \sum_{q=\pm 1} \gamma_q(P) \hat{V}$$



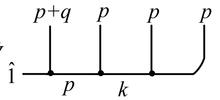
, (4)

and dealing with the free edge whose color 1 is denoted by $\hat{1}$ in the graph of expression (4) that the volume operator $\hat{V}$ does not operate to it, then in (4), the action of the operator $\hat{V}$ on the 5-valent vertex ϕ_{pk} may be written as [9]

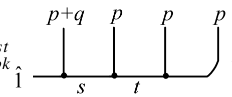
$$\hat{V} \phi_{pk} = \sum_{s,t} V_{pk}^{st} \phi_{st}, \tag{5a}$$

this is

$$N_{pk}(\hat{1}, p+q, p, p, p) \hat{V}$$



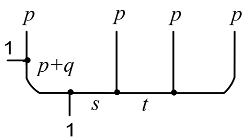
$$= \sum_{s,t} N_{st}(\hat{1}, p+q, p, p, p) V_{pk}^{st}$$



. (5b)

So expression (4) becomes

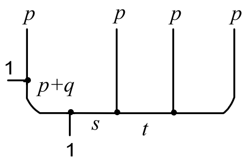
$$\hat{V} h_0^{-1} \phi_k = \sum_{q=\pm 1} \sum_{s,t} \gamma_q(p) N_k(p) \left(\frac{N_{st} V_{pk}^{st}}{N_{pk}} \right) (\hat{1}, p+q, p, p, p)$$



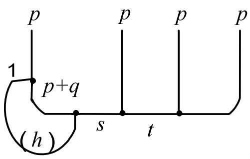
. (6)

For the operation of $\tau^i h_0$ on the last graph in (6), we have

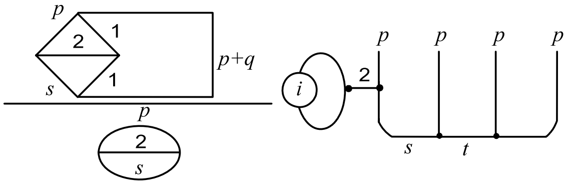
$$\tau^i h_0$$



$$= \tau^i$$



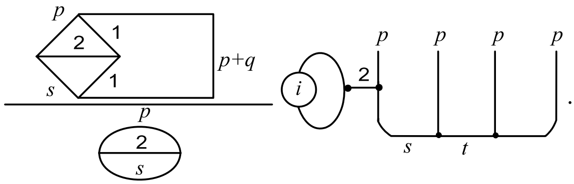
$$=$$



.

Introducing the above result into (2), we have

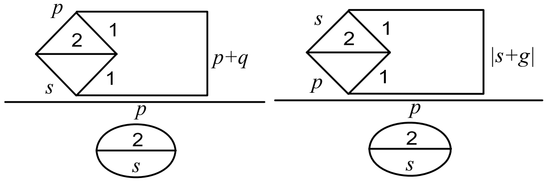
$$\hat{\Theta}_0^i\phi_k = \frac{4}{i\hbar k}\sum_{q=\pm 1}\sum_{s,t}\gamma_q(p)N_k(p)\left(\frac{N_{st}V_{pk}^{st}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)$$

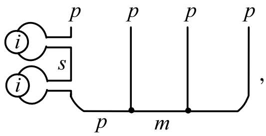
$\times$  . (7)

Exerting the operation of $\hat{\Theta}_0^i$ on the expression (7) again, and putting obtained result into (1), one has

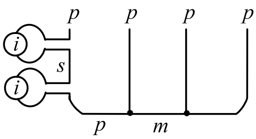
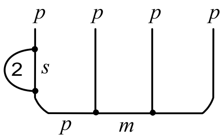
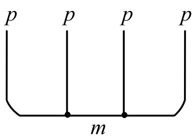
$$\hat{M}(s_0, s_0)\phi_k = -\frac{16}{(\hbar k)^2}\sum_{q=\pm 1}\sum_{s,t}\sum_{g=\pm 1}\sum_{m}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p)$$

$$\times\left(\frac{N_{st}V_{pk}^{st}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1}, |s+g|, p, p, p)$$

$\times$ 

$\times$  ,

where

 $= \dfrac{1}{2}$  $= \dfrac{1}{2}\dfrac{\theta(s,2,p)}{\Delta_p}$  .

Finally, the eigenaction of the metric component operator $\hat{M}(s_0, s_0)$ on the state ϕ_k is given

$$\hat{M}(s_0, s_0)\phi_k = \sum_m M(s_0, s_0)_{(p)km}\phi_m, \tag{8a}$$

where the matrix elements of the metric component is

$$M(s_0, s_0)_{(p)km} = -\frac{8}{(\hbar k)^2}\sum_{q=\pm 1}\sum_{s,t}\sum_{g=\pm 1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p)$$

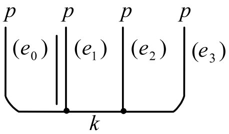
$$\times\left(\frac{N_{st}V_{pk}^{st}}{N_{pk}}\right)(\hat{1},p+q,p,p,p)\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p)$$

$$\times\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g|\\ 1 & 1 & 2\end{bmatrix}}{\Delta_p\theta(p,2,s)}. \tag{8b}$$

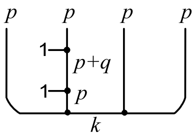
1.2 The Eigenactions of Other Diagonal Component Operators on ϕ_k

Similarly, the action of holonomy h_1^{-1} on the vertex $\phi_k(p,p,p,p)$ is

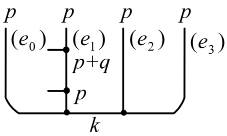
$$h_1^{-1}\phi_k(p,p,p,p)=N_k(p)$$



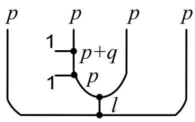
$$=N_k(p)\sum_{q=\pm1}\gamma_q(p)$$



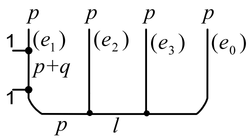
In order to operate the volume operator $\hat{V}$ on the above expression, by using of the recoupling theorem [10], the last graph in the expression need to become the following form:



$$=\sum_l\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}$$



$$=\sum_l\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}$$



$$. \tag{9}$$

The action of other factors in $\hat{\Theta}_1^i$ on the expression (9) is different from the action exerted from $\hat{\Theta}_0^i$ that, in the final result need to consider a 6-j symbol which shall appear in the computation about the 5-valent vertex $(\hat{1},p+q,p,p,p)$, hence, the action given by $\hat{M}(s_1,s_1)$ on ϕ_k is

$$\hat{M}(s_1,s_1)\phi_k=\sum_m M(s_1,s_1)_{(p)km}\phi_m,$$

here the matrix elements

$$M(s_1,s_1)_{(p)km}=-\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p)$$

$$\times\left(\sum_l\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{st}V_{pl}^{st}}{N_{pl}}\right)(\hat{1},p+q,p,p,p)\right)$$

$$\times\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1},|s+g|,p,p,p)$$

$$\times\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g|\\ 1 & 1 & 2\end{bmatrix}}{\Delta_p\theta(s,2,p)}. \tag{10}$$

Similarly, the actions of operators $\hat{M}(s_2, s_2)$ and $\hat{M}(s_3, s_3)$ on vertex ϕ_k can be obtained:

$$\hat{M}(s_2, s_2)\phi_k = \sum_m M(s_2, s_2)_{(p)km}\phi_m,$$

here

$$\begin{aligned} M(s_2, s_2)_{(p)km} = &-\frac{8}{(\hbar k)^2}\sum_{q=\pm 1}\sum_{s,t}\sum_{g=\pm 1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p) \\ &\times\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k\end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & l'\end{Bmatrix}\left(\frac{N_{st}V_{pl}^{st}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right) \\ &\times\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1}, |s+g|, p, p, p) \\ &\times\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_p\theta(s, 2, p)} \end{aligned} \tag{11}$$

and

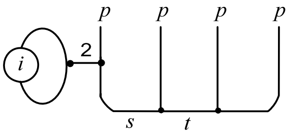
$$\hat{M}(s_3, s_3)\phi_k = \sum_m M(s_3, s_3)_{(p)km}\phi_m,$$

here

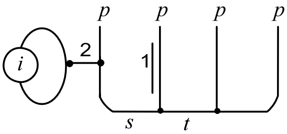
$$\begin{aligned} M(s_3, s_3)_{(p)km} = &-\frac{8}{(\hbar k)^2}\sum_{q=\pm 1}\sum_{s,t}\sum_{g=\pm 1}\gamma_q(p)\gamma_g(s)\left(\frac{N_k}{N_m}\right)(p) \\ &\times\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k\end{Bmatrix}\sum_{l''}\begin{Bmatrix} p & p & l'' \\ p & p & l'\end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & l''\end{Bmatrix}\right. \\ &\times\left.\left(\frac{N_{st}V_{pl}^{st}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\left(\frac{N_{pm}V_{st}^{pm}}{N_{st}}\right)(\hat{1}, |s+g|, p, p, p) \\ &\times\frac{\mathrm{Tet}\begin{bmatrix} s & p & p+q \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & s & |s+g| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_p\theta(s, 2, p)}. \end{aligned} \tag{12}$$

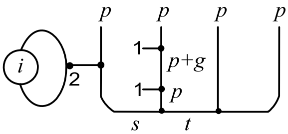
2 The Eigenactions of Metric Matrix Off-Diagonal Component Operators $\hat{M}(s_\alpha, s_\beta)$ on Vertex ϕ_k

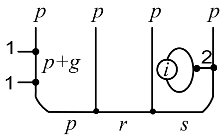
For the operation of operator $\hat{M}(s_\alpha, s_\beta)$ $(\alpha \neq \beta)$ on the vertex ϕ_k, because the two actions are exerted on different edges, the computation has some difference from that given in Sect. 1. For example, for the action of operator $\hat{M}(s_0, s_1)$ on 4-valent vertex ϕ_k, the action of $\hat{\Theta}_0^i$ on ϕ_k is same as the computation given by expression (7), whoever, the operation of $\hat{\Theta}_0^i$ on (7) is different from above. For this reason, let

$$F^i_{st} =$$ ,

then one has the operation

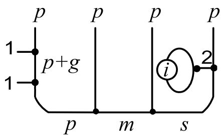
$$h_1^{-1} F^i_{st} =$$ 

$$= \sum_{g=\pm 1} \gamma_g(p)$$ 

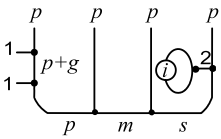
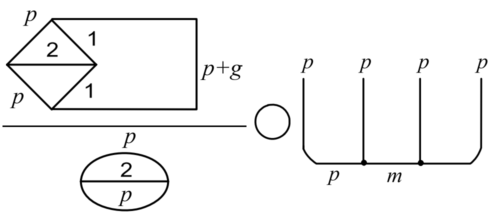
$$= \sum_{g=\pm 1} \gamma_g(p) \left(\sum_r \begin{Bmatrix} s & p & r \\ p & p & t \end{Bmatrix}\right.$$  $$\left.\vphantom{\sum_r}\right).$$

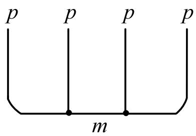
After the action of volume operator $\hat{V}$ on the above expression, it becomes

$$\hat{V} h_1^{-1} F^i_{st} = \sum_m \sum_{g=\pm 1} \gamma_g(p) \left(\sum_r \begin{Bmatrix} p & p & r \\ p & p & t \end{Bmatrix} \left(\frac{N_{pm} V^{pm}_{pr}}{N_{pr}} \right) (\hat{1}, p+g, p, p, p) \right)$$

$$\times$$  . (13)

In the r.h.s. of (13), the last graph under the action of $\tau^i h_1$ shall be changed as

$$\tau^i h_1$$  $$= \frac{1}{2}$$ 

$$= -\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+g \\ 1 & 1 & 2 \end{bmatrix}}{\theta(p, 2, p)}$$  . (14)

From (13, 14), and after rearrangement, we have

$$\begin{aligned}\hat{\Theta}_1^i\hat{\Theta}_0^i\phi_k = \frac{16}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\sum_m \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)\\ \times\left(\sum_r\begin{Bmatrix} p & p & r\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right)\\ \times\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}\phi_m.\end{aligned} \tag{15}$$

The result of action of $\hat{\Theta}_0^i\hat{\Theta}_1^i$ on ϕ_k can be similarly computed, it is

$$\begin{aligned}\hat{\Theta}_0^i\hat{\Theta}_1^i\phi_k = \frac{16}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t\sum_m \gamma_q(p)\gamma_g(p)\\ \times\left(\frac{N_k}{N_m}\right)(p)\left(\sum_l\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\\ \times\left(\sum_{l''}\begin{Bmatrix} p & p & l''\\ p & p & t\end{Bmatrix}\sum_{l'}\begin{Bmatrix} p & p & l'\\ p & p & l''\end{Bmatrix}\sum_l\begin{Bmatrix} p & p & l\\ p & p & l'\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)\right.\\ \left.\times(\hat{1}, p+g, p, p, p)\right)\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}\phi_m.\end{aligned}$$

Taking the above expression and (15) into (1), the result is

$$\hat{M}(s_0, s_1)\phi_k = \sum_m M(s_0, s_1)_{(p)km}\phi_m,$$

where the matrix elements

$$\begin{aligned}&M(s_0, s_1)_{(p)k_m}\\ &= \frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_t \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left\{\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)\right.\\ &\quad\times\left(\sum_r\begin{Bmatrix} p & p & r\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right)\\ &\quad+\left(\sum_l\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\\ &\quad\times\left(\sum_{l''}\begin{Bmatrix} p & p & l''\\ p & p & t\end{Bmatrix}\sum_{l'}\begin{Bmatrix} p & p & l'\\ p & p & l''\end{Bmatrix}\sum_l\begin{Bmatrix} p & p & l\\ p & p & l'\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)\right.\\ &\quad\left.\left.\times(\hat{1}, p+g, p, p, p)\right)\right\}\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}.\end{aligned} \tag{16}$$

For the eigenactions of $\hat{M}(s_0, s_2)$ and $\hat{M}(s_0, s_3)$, the similar computations shall give the following results:

$$\hat{M}(s_0, s_2)\phi_k = \sum_m M(s_0, s_2)_{(p)km}\phi_m,$$

where

$$\begin{aligned}
&M(s_0, s_2)_{(p)km} \\
&\quad = \frac{8}{(\hbar k)^2} \sum_{q=\pm 1} \sum_{g=\pm 1} \sum_t \gamma_q(p)\gamma_q(s)\left(\frac{N_k}{N_m}\right)(p) \\
&\qquad \times \left\{\left(\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)\left(\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & t \end{Bmatrix}\sum_r \begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\right.\right. \\
&\qquad \times \left.\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right) + \left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_l \begin{Bmatrix} p & p & l \\ p & p & l' \end{Bmatrix}\right. \\
&\qquad \times \left.\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & t \end{Bmatrix}\sum_l \begin{Bmatrix} p & p & l \\ p & p & l' \end{Bmatrix}\right. \\
&\qquad \times \left.\left.\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1}, p+g, p, p, p)\right)\right\} \frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g \\ 1 & 1 & 2 \end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}
\end{aligned} \tag{17}$$

and

$$\hat{M}(s_0, s_3)\phi_k = \sum_m M(s_0, s_3)_{(p)km}\phi_m,$$

where

$$\begin{aligned}
&M(s_0, s_3)_{(p)km} \\
&\quad = \frac{8}{(\hbar k)^2} \sum_{q=\pm 1} \sum_{g=\pm 1} \sum_t \gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left\{\left(\frac{N_{pt}V_{pk}^{st}}{N_{pk}}\right)(\hat{1}, p+q, p, p, p)\right. \\
&\qquad \times \left(\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & t \end{Bmatrix}\sum_{r''}\begin{Bmatrix} p & p & r'' \\ p & p & r' \end{Bmatrix}\sum_r \begin{Bmatrix} p & p & r \\ p & p & r'' \end{Bmatrix}\right. \\
&\qquad \times \left.\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right) \\
&\qquad + \left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l''}\begin{Bmatrix} p & p & l'' \\ p & p & l' \end{Bmatrix}\sum_l \begin{Bmatrix} p & p & l \\ p & p & l'' \end{Bmatrix}\right. \\
&\qquad \times \left.\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)
\end{aligned}$$

$$\times\left(\sum_{l}\begin{Bmatrix} p & p & l\\ p & p & t\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1}, p+g, p, p, p)\right)\Bigg\}$$

$$\times\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}. \tag{18}$$

The action of operators $\hat{M}(s_1, s_2)$, $\hat{M}(s_1, s_3)$ and $\hat{M}(s_2, s_3)$ on the vertex ϕ_k, may be also calculated by using the similar operations as above, we shall give the results of them directly:

$$\hat{M}(s_1, s_2)\phi_k = \sum_m M(s_1, s_2)_{(p)km}\phi_m,$$

where

$$\begin{aligned}
&M(s_1, s_2)_{(p)km}\\
&= \frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_{t}\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\\
&\times\Bigg\{\left(\sum_{l}\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\left(\sum_{r}\begin{Bmatrix} p & p & r\\ p & p & t\end{Bmatrix}\right.\\
&\times\left.\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right)+\left(\sum_{l'}\begin{Bmatrix} p & p & l'\\ p & p & k\end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l\\ p & p & l'\end{Bmatrix}\right.\\
&\times\left.\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\left(\sum_{l''}\begin{Bmatrix} p & p & l''\\ p & p & t\end{Bmatrix}\sum_{l'}\begin{Bmatrix} p & p & l'\\ p & p & l''\end{Bmatrix}\right.\\
&\times\left.\sum_{l}\begin{Bmatrix} p & p & l\\ p & p & l'\end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1}, p+g, p, p, p)\right)\Bigg\}\\
&\times\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g\\ 1 & 1 & 2\end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}
\end{aligned} \tag{19}$$

and

$$\hat{M}(s_1, s_3)\phi_k = \sum_m M(s_1, s_3)_{(p)km}\phi_m,$$

where

$$\begin{aligned}
&M(s_1, s_2)_{(p)km}\\
&= \frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{g=\pm1}\sum_{t}\gamma_q(q)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\\
&\times\Bigg\{\left(\sum_{l}\begin{Bmatrix} p & p & l\\ p & p & k\end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & t \end{Bmatrix}\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right) \\
&+\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l''}\begin{Bmatrix} p & p & l'' \\ p & p & l' \end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & l'' \end{Bmatrix}\right. \\
&\left.\times\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right) \\
&\left.\times\left(\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & t \end{Bmatrix}\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1}, p+g, p, p, p)\right)\right\} \\
&\times\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g \\ 1 & 1 & 2 \end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}
\end{aligned}
\tag{20}
$$

and

$$
\hat{M}(s_2, s_3)\phi_k = \sum_m M(s_2, s_3)_{(p)km}\phi_m,
$$

where

$$
\begin{aligned}
&M(s_2, s_3)_{(p)km} \\
&\quad= \frac{8}{(\hbar k)^2}\sum_{q=\pm 1}\sum_{g=\pm 1}\sum_{t}\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p) \\
&\quad\times\left\{\left(\sum_{l}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l'}\begin{Bmatrix} p & p & l \\ p & p & l' \end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right)\right. \\
&\quad\times\left(\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & t \end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right) \\
&\quad+\left(\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & p & k \end{Bmatrix}\sum_{l''}\begin{Bmatrix} p & p & l'' \\ p & p & l' \end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & l'' \end{Bmatrix}\right. \\
&\quad\left.\times\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right) \\
&\quad\times\left(\sum_{r''}\begin{Bmatrix} p & p & r'' \\ p & p & t \end{Bmatrix}\sum_{r'}\begin{Bmatrix} p & p & r' \\ p & p & r'' \end{Bmatrix}\sum_{r}\begin{Bmatrix} p & p & r \\ p & p & r' \end{Bmatrix}\right. \\
&\quad\left.\left.\times\left(\frac{N_{pm}V_{pr}^{n\,pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right)\right\} \\
&\quad\times\frac{\mathrm{Tet}\begin{bmatrix} p & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+g \\ 1 & 1 & 2 \end{bmatrix}}{\theta(p,2,p)\theta(p,2,p)}.
\end{aligned}
\tag{21}
$$

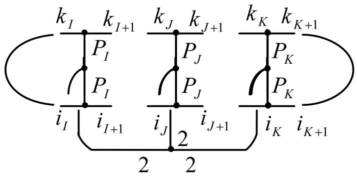
3 Expression of Generic Recoupling Matrix for Volume Operator

The action of operator $\hat{V}$ on 5-valent vertex ϕ_{pk} is operated with (5), the volume matrix elements V_{pk}^{st} in (5b) are given by

$$V_{pk}^{st} = \frac{(\hbar k)^{\frac{3}{2}}}{4}\sqrt{\sum_{[IJK]} W_{[IJK]_{pk}}^{st}}, \tag{22}$$

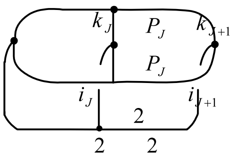
where $W_{[IJK]_{pk}}^{st}$ are the recoupling matrix elements. For the calculation of the recoupling matrices, we here first introduce a generic formula used to any-valent vertex calculation [9]:

$$W_{[IJK]_{\vec{i}}}^{\vec{k}} = N_{\vec{i}} N_{\vec{k}} P_I P_J P_K \cdot p_0$$



$$p_{n-1}$$

$$= N_{\vec{i}} N_{\vec{k}} P_I P_J P_K \alpha_i \alpha_{ii} \alpha_{iii} \alpha_{iv}$$



$$= N_{\vec{i}} N_{\vec{k}} P_I P_J P_K \alpha_i \alpha_{ii} \alpha_{iii} \alpha_{iv} \begin{Bmatrix} k_J & P_J & k_{J+1} \\ i_J & P_J & i_{J+1} \\ 2 & 2 & 2 \end{Bmatrix}, \tag{23}$$

where $N_{\vec{i}}$, $N_{\vec{k}}$ are normalization factors, $\vec{i} = i_2, \ldots, i_{n-2}$, $\vec{k} = k_2, \ldots, \vec{k}_{n-2}$. $\alpha_i, \alpha_{ii}, \alpha_{iii}, \alpha_{iv}$ are given as follows:

$$\alpha_i = -\lambda_{k_{I+1}}^{i_{I+1}2} \left(\prod_{x=2}^{I} \delta_{i_x}^{k_x}\right) \frac{\theta(p_0, p_1, i_2)}{\Delta_{i_2}} \left(\prod_{x=2}^{I-1} \frac{\theta(i_x, p_x, i_{x+1})}{\Delta_{i_{2+1}}}\right) \frac{\mathrm{Tet}\begin{bmatrix} k_{I+1} & i_{I+1} & i_I \\ p_I & p_I & 2 \end{bmatrix}}{\theta(k_{I+1}, i_{I+1}, 2)}, \tag{24}$$

$$\alpha_{ii} = \prod_{x=I+1}^{J-1} \frac{\mathrm{Tet}\begin{bmatrix} k_x & k_{x+1} & 2 \\ i_{x+1} & i_x & p_x \end{bmatrix}}{\theta(k_{x+1}, i_{x+1}, 2)}, \tag{25}$$

$$\alpha_{iii} = \frac{\lambda_{k_K}^{i_K 2}}{\lambda_{k_{J+1}}^{i_{J+1}2}} \left(\prod_{x=J+1}^{K-1} \frac{\mathrm{Tet}\begin{bmatrix} k_x & k_{x+1} & 2 \\ i_{x+1} & i_x & p_x \end{bmatrix}}{\theta(k_x, i_x, 2)}\right), \tag{26}$$

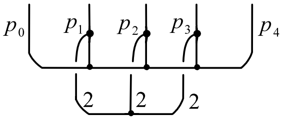
$$\alpha_{iv} = \left(\prod_{x=K+1}^{n-2} \delta_{i_x}^{k_x}\right)\left(\prod_{x=K+1}^{n-3} \frac{\theta(i_x, p_x, i_{x+1})}{\Delta_{i_x}}\right) \frac{\theta(i_{n-2}, p_{n-2}, p_{n-1})}{\Delta_{i_{n-2}}} \frac{\mathrm{Tet}\begin{bmatrix} i_k & k_k & i_{k+1} \\ p_k & p_k & 2 \end{bmatrix}}{\theta(k_k, i_k, 2)}. \tag{27}$$

4 Recoupling Matrices Used to the Calculation of Generic 5-Valent Vertex

In the paper the volume operator $\hat{V}$ shall be operated only on 5-valent vertex, and the little edge whose color is $\hat{1}$ in the vertex does not be visited by $\hat{V}$, so the volume operator $\hat{V}$ has mere four hand triples to grasping the edges of the generic 5-valent vertex, that is, the triples "123", "124", "134" and "234". Below we shall compute the recoupling matrices corresponding to the four hand triples.

4.1 $W^{\vec{k}}_{[123]\vec{i}}$

In this case, since $I=1$, $J=2$, $K=3$, the grasping graph in expression (23) becomes



, (28)

and $\alpha_{ii}=\alpha_{iii}=1$, $\vec{i}=i_1 i_2$, $\vec{k}=k_1 k_2$. So from (23) we have

$$W^{\vec{k}}_{[123]\vec{i}} = N_{\vec{i}} N_{\vec{k}} P_1 P_2 P_3 \alpha_i \alpha_{iv} \begin{Bmatrix} k_2 & p_2 & k_3 \\ i_2 & p_2 & i_3 \\ 2 & 2 & 2 \end{Bmatrix}, \tag{29}$$

where

$$N_{\vec{k}}(p_0,\ldots,p_4) = \sqrt{\frac{\prod_{x=2}^{3}\Delta_{k_x}}{\prod_{x=1}^{3}\theta(k_x,p_x,k_{x+1})}}, \tag{30}$$

$$N_{\vec{i}}(p_0,\ldots,p_4) = \sqrt{\frac{\prod_{x=2}^{3}\Delta_{i_x}}{\prod_{x=1}^{3}\theta(i_x,p_x,i_{x+1})}}, \tag{31}$$

$$\alpha_i = -\lambda^{i_2 2}_{k_2} \frac{\mathrm{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2,2,k_2)} = -\frac{\mathrm{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2,2,k_2)}\lambda^{i_2 2}_{k_2}, \tag{32}$$

$$\alpha_{iv} = \frac{\mathrm{Tet}\begin{bmatrix} i_3 & k_3 & i_4 \\ p_3 & p_3 & 2 \end{bmatrix}}{\theta(k_3,2,i_3)}. \tag{33}$$

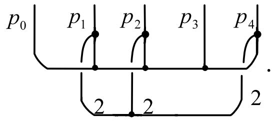
Introducing (30–33) into (29), the recoupling matrix for the triple [123] is got:

$$W^{\vec{k}}_{[123]\vec{i}} = \sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1,p_1,i_2)\theta(i_2,p_2,i_3)\theta(i_3,p_3,i_4)\theta(k_1,p_1,k_2)\theta(k_2,p_2,k_3)\theta(k_3,p_3,k_4)}}$$
$$\times(-1)\lambda^{i_2 2}_{k_2} p_1 p_2 p_2 \frac{\mathrm{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2,2,k_2)} \frac{\mathrm{Tet}\begin{bmatrix} i_3 & k_3 & i_4 \\ p_3 & p_3 & 2 \end{bmatrix}}{\theta(i_3,2,k_3)} \begin{Bmatrix} k_2 & p_2 & k_3 \\ i_2 & p_2 & i_3 \\ 2 & 2 & 2 \end{Bmatrix}, \tag{34}$$

where $i_1=k_1=p_0$, $i_4=k_4=p_4$.

4.2 $W^{\vec{k}}_{[124]\vec{i}}$

For this case, $I = 1$, $J = 2$, $K = 4$, the grasping graph of hand triple is



Since $\alpha_{ii} = \alpha_{iv} = 1$, the recoupling matrix for the hand triple [124] becomes

$$W^{\vec{k}}_{[124]\vec{i}} = N_{\vec{i}} N_{\vec{k}} P_1 P_2 P_4 \alpha_i \alpha_{iii} \begin{Bmatrix} k_2 & p_2 & k_3 \\ i_2 & p_2 & i_3 \\ 2 & 2 & 2 \end{Bmatrix}, \tag{35}$$

where

$$\alpha_i = \frac{\text{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2, 2, k_2)} (-\lambda^{i_2 2}_{k_2}), \tag{36}$$

$$\alpha_{iii} = \frac{\lambda^{i_4 2}_{k_4}}{\lambda^{i_3 2}_{k_3}} \frac{\text{Tet}\begin{bmatrix} k_3 & k_4 & 2 \\ i_4 & i_3 & p_3 \end{bmatrix}}{\theta(k_3, i_3, 2)} = \frac{\text{Tet}\begin{bmatrix} k_3 & k_4 & 2 \\ i_4 & i_3 & p_3 \end{bmatrix}}{\theta(k_3, i_3, 2)} \frac{-1}{\lambda^{i_3 2}_{k_3}}. \tag{37}$$

Putting (30), (31), (36) and (37) into (35), we have

$$W^{\vec{k}}_{[124]\vec{i}} = \sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1, p_1, i_2)\theta(i_2, p_2, i_3)\theta(i_3, p_3, i_4)\theta(k_1, p_1, k_2)\theta(k_2, p_2, k_3)\theta(k_3, p_3, k_4)}}$$
$$\times \frac{(-1)^2 \lambda^{i_t 2}_{k_2}}{\lambda^{i_3 2}_{k_3}} p_1 p_2 p_4 \frac{\text{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2, 2, k_2)} \frac{\text{Tet}\begin{bmatrix} k_3 & k_4 & 2 \\ i_4 & i_3 & p_3 \end{bmatrix}}{\theta(k_3, i_3, 2)} \begin{Bmatrix} k_2 & p_2 & k_3 \\ i_2 & p_2 & i_3 \\ 2 & 2 & 2 \end{Bmatrix}. \tag{38}$$

4.3 $W^{\vec{k}}_{[234]\vec{i}}$ and $W^{\vec{k}}_{[134]\vec{i}}$

Through similar calculations, their results may be obtained. For simplicity we directly give the results:

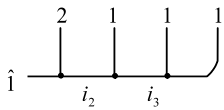
$$W^{\vec{k}}_{[234]\vec{i}} = \sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1, p_1, i_2)\theta(i_2, p_2, i_3)\theta(i_3, p_3, i_4)\theta(k_1, p_1, k_2)\theta(k_2, p_2, k_3)\theta(k_3, p_3, k_4)}}$$
$$\times p_2 p_3 p_4 \delta^{k_2}_{i_2} \frac{\theta(p_0, p_1, i_2)}{\Delta_{i_2}} \frac{\text{Tet}\begin{bmatrix} k_3 & i_3 & i_2 \\ p_2 & p_2 & 2 \end{bmatrix}}{\theta(k_3, i_3, 2)} \begin{Bmatrix} k_3 & p_3 & k_4 \\ i_3 & p_3 & i_4 \\ 2 & 2 & 2 \end{Bmatrix} (-1)\lambda^{i_3 2}_{k_3}, \tag{39}$$

and

$$W_{[134]_{\vec{i}}}^{\vec{k}} = \sqrt{\frac{\Delta_{i_2}\Delta_{i_3}\Delta_{k_2}\Delta_{k_3}}{\theta(i_1,p_1,i_2)\theta(i_2,p_2,i_3)\theta(i_3,p_3,i_4)\theta(k_1,p_1,k_2)\theta(k_2,p_2,k_3)\theta(k_3,p_3,k_4)}}$$
$$\times\, p_1p_3p_4(-1)\lambda_{k_2}^{i_2 2}\frac{\mathrm{Tet}\begin{bmatrix} k_2 & i_2 & i_1 \\ p_1 & p_1 & 2 \end{bmatrix}}{\theta(i_2,2,k_2)}\frac{\mathrm{Tet}\begin{bmatrix} k_2 & k_3 & 2 \\ i_3 & i_2 & p_2 \end{bmatrix}}{\theta(k_3,i_3,2)}\begin{Bmatrix} k_3 & p_3 & k_4 \\ i_3 & p_3 & i_4 \\ 2 & 2 & 2 \end{Bmatrix}. \tag{40}$$

5 Recoupling Matrix Elements for 5-Valent Vertex $(\hat{1}, 2, 1, 1, 1)$

In this paper we shall give the calculations about Gaussian weave state of $p = 1$, at present we compute the case of $k = 0$ (if $p = 1$, only have $k = 0, 2$), so the 5-valent vertices $(\hat{1}, p+q, 1, 1, 1)$ and $(\hat{1}, |s+g|, 1, 1, 1)$ could have mere three forms $(\hat{1}, 0, 1, 1, 1)$, $(\hat{1}, 2, 1, 1, 1)$ and $(\hat{1}, 4, 1, 1, 1)$. Since the contribution of the last vertex $(\hat{1}, 4, 1, 1, 1)$ under the action of volume operator $\hat{V}$ does not exist, so we need only to compute the recoupling matrices with respect to vertices $(\hat{1}, 2, 1, 1, 1)$ and $(\hat{1}, 0, 1, 1, 1)$. For the vertex $(\hat{1}, 2, 1, 1, 1)$, its graph is



. (41)

5.1 Matrix Elements of Recoupling Matrix $W_{[123]_{i_2 i_3}}^{k_2 k_3}$

Because the duple color of the virtual edges $i_2 i_3$ in (41) may be 10, 12 and 32, employing (34) through calculation, the following recoupling matrix elements are obtained:

$$W_{[123]_{10}}^{10} = \sqrt{\frac{\Delta_1\Delta_0\Delta_1\Delta_0}{\theta(1,2,1)\theta(1,1,0)\theta(0,1,1)\theta(1,2,1)\theta(1,1,0)\theta(0,1,1)}}(-1)^2$$
$$\times\, 2\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(0,2,0)}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{Bmatrix} = 0,$$

$$W_{[123]_{10}}^{12} = \frac{\sqrt{\Delta_1\Delta_2\Delta_1\Delta_0}}{|\theta(1,2,1)\theta(1,1,2)\theta(1,1,0)|}$$
$$\times\, 2\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(2,2,0)}(-1)^2\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{Bmatrix}$$
$$= \frac{2}{3}\sqrt{3},$$

$$
\begin{aligned}
W^{32}_{[123]_{10}} &= \frac{\sqrt{\Delta_1\Delta_0\Delta_3\Delta_2}}{|\theta(1,2,1)\theta(1,1,0)\theta(1,2,3)|} \\
&\quad \times 2\frac{\mathrm{Tet}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(1,2,3)} \frac{\mathrm{Tet}\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(2,2,0)} (-1)^2 \begin{Bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{Bmatrix} \\
&= -\frac{\sqrt{6}}{6}, \\
W^{10}_{[123]_{12}} &= \frac{\sqrt{\Delta_1\Delta_2\Delta_1\Delta_0}}{|\theta(1,2,1)\theta(1,1,2)\theta(1,1,0)|} \\
&\quad \times 2\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(1,2,1)} \frac{\mathrm{Tet}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(0,2,2)} (-1)^2 \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{Bmatrix} \\
&= -\frac{2}{3}\sqrt{3}, \\
W^{12}_{[123]_{12}} &= \frac{\sqrt{\Delta_1\Delta_2\Delta_1\Delta_2}}{|\theta(1,2,1)\theta(1,1,2)\theta(2,1,1)|} \\
&\quad \times 2\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(1,2,1)} \frac{\mathrm{Tet}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(2,2,2)} (-1)^2 \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{Bmatrix} \\
&= 0, \\
W^{32}_{[123]_{12}} &= \frac{\sqrt{\Delta_3\Delta_2\Delta_1\Delta_2}}{|\theta(1,1,2)\theta(1,1,2)\theta(1,2,3)|} \\
&\quad \times 2\frac{\mathrm{Tet}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(1,2,3)} \frac{\mathrm{Tet}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(2,2,2)} \begin{Bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{Bmatrix} (-1)^2 \\
&= -\frac{\sqrt{2}}{2}, \\
W^{10}_{[123]_{32}} &= \frac{\sqrt{\Delta_1\Delta_0\Delta_3\Delta_2}}{|\theta(1,1,2)\theta(1,1,0)\theta(1,2,3)|} \\
&\quad \times 2\frac{\mathrm{Tet}\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \end{bmatrix}}{\theta(3,2,1)} \frac{\mathrm{Tet}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(0,2,2)} (-1)^2 \begin{Bmatrix} 1 & 1 & 0 \\ 3 & 1 & 2 \\ 2 & 2 & 2 \end{Bmatrix} \\
&= \frac{\sqrt{6}}{6},
\end{aligned}
$$

$$W_{[123]_{32}}^{12} = \frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_2}}{|\theta(1,1,2)\theta(1,1,2)\theta(1,2,3)|}$$

$$\times 2\frac{\mathrm{Tet}\begin{bmatrix}1 & 3 & 1\\ 2 & 2 & 2\end{bmatrix}}{\theta(3,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}2 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}}{\theta(2,2,2)}(-1)^2\begin{Bmatrix}1 & 1 & 2\\ 3 & 1 & 2\\ 2 & 2 & 2\end{Bmatrix}$$

$$= \frac{\sqrt{2}}{2},$$

$$W_{[123]_{32}}^{32} = \frac{\sqrt{\Delta_3\Delta_2\Delta_3\Delta_2}}{|\theta(1,2,3)\theta(1,2,3)\theta(1,1,2)|}$$

$$\times 2\frac{\mathrm{Tet}\begin{bmatrix}3 & 3 & 1\\ 2 & 2 & 2\end{bmatrix}}{\theta(3,2,3)}\frac{\mathrm{Tet}\begin{bmatrix}2 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}}{\theta(2,2,2)}(-1)^2\begin{Bmatrix}3 & 1 & 2\\ 3 & 1 & 2\\ 2 & 2 & 2\end{Bmatrix}$$

$$= 0.$$

Hence the corresponding recoupling matrix may be written as

$$W_{[123]_{i_2 i_3}}^{k_2 k_3} = \begin{bmatrix} W_{[123]_{10}}^{10} & W_{[123]_{10}}^{12} & W_{[123]_{10}}^{32} \\ W_{[123]_{12}}^{10} & W_{[123]_{12}}^{12} & W_{[123]_{12}}^{32} \\ W_{[123]_{32}}^{10} & W_{[123]_{32}}^{12} & W_{[123]_{32}}^{32} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ -\frac{2}{3}\sqrt{3} & 0 & -\sqrt{2}/2 \\ \sqrt{6}/6 & \sqrt{2}/2 & 0 \end{bmatrix}. \tag{42}$$

5.2 Matrix Elements of Other Recoupling Matrices

On a similar plan to obtaining the result (42), the other recoupling matrices may be got:

$$W_{[124]_{i_2 i_3}}^{k_2 k_3} = \begin{bmatrix} W_{[124]_{10}}^{10} & W_{[124]_{10}}^{12} & W_{[124]_{10}}^{32} \\ W_{[124]_{12}}^{10} & W_{[124]_{12}}^{12} & W_{[124]_{12}}^{32} \\ W_{[124]_{32}}^{10} & W_{[124]_{32}}^{12} & W_{[124]_{32}}^{32} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3}\sqrt{3} & -\frac{\sqrt{6}}{6} \\ -\frac{2}{3}\sqrt{3} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \tag{43}$$

$$W_{[234]_{i_2 i_3}}^{k_2 k_3} = \begin{bmatrix} W_{[234]_{10}}^{10} & W_{[234]_{10}}^{12} & W_{[234]_{10}}^{32} \\ W_{[234]_{12}}^{10} & W_{[234]_{12}}^{12} & W_{[234]_{12}}^{32} \\ W_{[234]_{32}}^{10} & W_{[234]_{32}}^{12} & W_{[234]_{32}}^{32} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{44}$$

$$W_{[134]_{i_2 i_3}}^{k_2 k_3} = \begin{bmatrix} W_{[134]_{10}}^{10} & W_{[134]_{10}}^{12} & W_{[134]_{10}}^{32} \\ W_{[134]_{12}}^{10} & W_{[134]_{12}}^{12} & W_{[134]_{12}}^{32} \\ W_{[134]_{32}}^{10} & W_{[134]_{32}}^{12} & W_{[134]_{32}}^{32} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2}{3}\sqrt{3} & -\frac{\sqrt{6}}{3} \\ \frac{2}{3}\sqrt{3} & 0 & 0 \\ \frac{\sqrt{6}}{3} & 0 & 0 \end{bmatrix}. \tag{45}$$

5.3 Summary

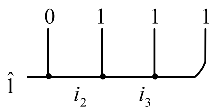
For the 5-valent vertex $(\hat{1}, 2, 1, 1, 1)$ under the action of volume operator $\hat{V}$, collecting the results (42–45) the four recoupling matrices may be given in the Table 1.

Table 1 Recoupling matrices for vertex $(\hat{1}, 2, 1, 1, 1)$

Element	Matrix			
	$W_{[123]}$	$W_{[124]}$	$W_{[234]}$	$W_{[134]}$
10 10	0	0	0	0
12 10	$\frac{2}{3}\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	$-\sqrt{3}/2$	$-\frac{2}{3}\sqrt{3}$
32 10	$-\frac{\sqrt{6}}{6}$	$-\sqrt{6}/6$	0	$-\sqrt{6}/3$
10 12	$-\frac{2}{3}\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	$\sqrt{3}/2$	$\frac{2}{3}\sqrt{3}$
12 12	0	0	0	0
32 12	$-\sqrt{2}/2$	$-\sqrt{2}/2$	0	0
10 32	$\sqrt{6}/6$	$\sqrt{6}/6$	0	$\sqrt{6}/3$
12 32	$\sqrt{2}/2$	$\sqrt{2}/2$	0	0
32 32	0	0	0	0

6 Recoupling Matrix Elements for 5-Valent Vertex $(\hat{1}, 0, 1, 1, 1)$

The vertex graph for the vertex is



, (46)

the duple virtual color $i_2 i_3$ has two combination values of 10 and 12, a similar computation gives the results as follows:

$$W^{k_1k_2}_{[123]_{i_1i_2}} = \begin{bmatrix} W^{10}_{[123]_{10}} & W^{12}_{[123]_{10}} \\ W^{10}_{[123]_{12}} & W^{12}_{[123]_{12}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{47}$$

$$W^{k_2k_3}_{[124]_{i_2i_3}} = \begin{bmatrix} W^{10}_{[124]_{10}} & W^{12}_{[124]_{10}} \\ W^{10}_{[124]_{12}} & W^{12}_{[124]_{12}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{48}$$

$$W^{k_2k_3}_{[134]_{i_2i_3}} = \begin{bmatrix} W^{10}_{[134]_{10}} & W^{12}_{[134]_{10}} \\ W^{10}_{[134]_{12}} & W^{12}_{[134]_{12}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{49}$$

$$W^{k_2k_3}_{[234]_{i_2i_3}} = \begin{bmatrix} W^{10}_{[234]_{10}} & W^{12}_{[234]_{10}} \\ W^{10}_{[234]_{12}} & W^{12}_{[234]_{12}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}. \tag{50}$$

Table 2 Recoupling matrices for vertex $(\hat{1}, 0, 1, 1, 1)$

Element	Matrix			
	$W_{[123]}$	$W_{[124]}$	$W_{[234]}$	$W_{[134]}$
10 10	0	0	0	0
12 10	0	0	$\sqrt{3}/2$	0
10 12	0	0	$-\sqrt{3}/2$	0
12 12	0	0	0	0

The four recoupling matrices with respect to the vertex $(\hat{1}, 0, 1, 1, 1)$ may be also got from (47–50) (Table 2).

7 Matrix Elements of Expectation Values of Volume Operator $\hat{V}$ and Normalization Factors

Putting the matrix elements given in Table 1 into the formula (22), the expectation value matrix elements of operator $\hat{V}$ on the vertex $(\hat{1}, 2, 1, 1, 1)$ shall be got:

$$V_{i_2 i_3}^{k_2 k_3} = \begin{bmatrix} V_{10}^{10} & V_{10}^{12} & V_{10}^{32} \\ V_{12}^{10} & V_{12}^{12} & V_{12}^{32} \\ V_{32}^{10} & V_{32}^{12} & V_{32}^{32} \end{bmatrix} = \frac{(\hbar k)^{3/2}}{4} \begin{bmatrix} 0 & \sqrt{\frac{5}{2}\sqrt{3}} & \sqrt{\frac{2}{3}\sqrt{6}} \\ \sqrt{\frac{5}{2}\sqrt{3}} & 0 & \sqrt[4]{2} \\ \sqrt{\frac{2}{3}\sqrt{6}} & \sqrt[4]{2} & 0 \end{bmatrix}. \tag{51}$$

For the vertex $(\hat{1}, 0, 1, 1, 1)$, the matrix elements of operator $\hat{V}$ may be similarly obtained:

$$V_{i_2 i_3}^{k_2 k_3} = \frac{(\hbar k)^{3/2}}{4} \begin{bmatrix} 0 & \sqrt{\frac{\sqrt{3}}{2}} \\ \sqrt{\frac{\sqrt{3}}{2}} & 0 \end{bmatrix}. \tag{52}$$

Recalling the formula (30) for the vertex $(\hat{1}, 2, 1, 1, 1)$, the following normalization factors are computed:

$$N_{10} = \frac{\sqrt{-6}}{6}, \qquad N_{12} = \frac{\sqrt{-2}}{3}, \qquad N_{32} = \frac{\sqrt{-1}}{2}. \tag{53}$$

For vertex $(\hat{1}, 0, 1, 1, 1)$, they are

$$N_{10} = \frac{1}{2}, \qquad N_{12} = \frac{\sqrt{3}}{3}. \tag{54}$$

8 Expectation Values of Metric Matrix Diagonal Component Operators

We first compute the expectation value of the metric component "00" ($\alpha = \beta = 0$). In the eigenequation (8a), since $k = 0$, we have $m = 0$, so the metric component may be simply written as

$$
\begin{aligned}
M(S_0,S_0) = {} & -\frac{8}{(\hbar k)^2}\sum_{q=\pm1}\sum_{s,t}\sum_{g=\pm1}\gamma_q(1)\gamma_g(s)\left(\frac{N_0}{N_0}\right)(1) \\
& \times\left(\frac{N_{st}V_{10}^{st}}{N_{10}}\right)(\hat{1},1+q,1,1,1)\left(\frac{N_{10}V_{st}^{10}}{N_{st}}\right)(\hat{1},|s+g|,1,1,1) \\
& \times\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 1+q \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s+g| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)},
\end{aligned}
\tag{55}
$$

here, from the compatibility condition satisfied by 3-valent vertex, we know that $s=1$ or 3. When $s=1$, then $t=0$ or 2; when $s=3$, $t=2$. Expending (55) with $\sum_{q=\pm1}$ and $\sum_{g=\pm1}$, we have

$$
\begin{aligned}
M(S_0,S_0) = {} & -\frac{8}{(\hbar k)^2}\sum_{s,t}\Bigg[\gamma_+(1)\gamma_+(s)\left(\frac{N_{st}V_{10}^{st}}{N_{10}}\right)(\hat{1},2,1,1,1) \\
& \times\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s+1|,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 2 \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s+1| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)} \\
& +\gamma_+(1)\gamma_-(s)\frac{N_{st}V_{10}^{st}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s-1|,1,1,1) \\
& \times\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 2 \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s-1| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)}+\gamma_-(1)\gamma_+(s) \\
& \times\frac{N_{st}V_{10}^{st}}{N_{10}}(\hat{1},0,1,1,1)\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s+1|,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 0 \\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)} \\
& \times\mathrm{Tet}\begin{bmatrix} 1 & s & |s+1| \\ 1 & 1 & 2\end{bmatrix}+\gamma_-(1)\gamma_-(s)\frac{N_{st}V_{10}^{st}}{N_{10}}(\hat{1},0,1,1,1) \\
& \times\frac{N_{10}V_{st}^{10}}{N_{st}}(\hat{1},|s-1|,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} s & 1 & 0 \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & s & |s-1| \\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(s,1,2)}\Bigg].
\end{aligned}
\tag{56}
$$

Let the four terms in the brackets of (56) be A, B, C, D respectively, and introducing the normalization factors and the volume matrix elements used to vertices $(\hat{1},2,1,1,1)$ and $(\hat{1},0,1,1,1)$ into them, the following results are obtained:

$$
\begin{aligned}
A = {} & \gamma_+(1)\gamma_+(s)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1) \\
& \times\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2\end{bmatrix}}{\Delta_1\theta(1,1,2)}+\gamma_+(1)\gamma_+(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(2,1,2)} + \gamma_+(1)\gamma_+(3)\\
&\times \frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},4,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}3&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&3&4\\1&1&2\end{bmatrix}}{\Delta_1\theta(3,1,2)}\\
&= 0 + \frac{\frac{\sqrt{-2}}{3}}{\frac{\sqrt{-6}}{6}}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}\frac{\sqrt{-6}/6}{\sqrt{-2}/3}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}\frac{\frac{3}{2}}{(-2)3}\frac{3}{2} + 0\\
&= -\frac{15\sqrt{3}}{256}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
B &= \gamma_+(1)\gamma_+(1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\\
&\quad\times \frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_+(1)\gamma_-(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\\
&\quad\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_+(1)\gamma_-(3)\\
&\quad\times \frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},2,1,1,1)\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}3&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&3&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(3,1,2)}\\
&= 0 + \left(-\frac{1}{2}\right)\frac{\frac{\sqrt{-2}}{3}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}}{\sqrt{-6}/6}\frac{\frac{1}{2}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\sqrt{3}/3}\frac{\frac{3}{2}3}{(-2)3} + \left(-\frac{3}{4}\right)\\
&\quad\times \frac{\frac{\sqrt{-1}}{2}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{2}{3}\sqrt{6}}}{\sqrt{-6}/6}\frac{\frac{\sqrt{-6}}{6}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{2}{3}\sqrt{6}}}{\sqrt{-1}/2}\frac{(-4)(-4)}{(-2)(-4)}\\
&= \frac{3\sqrt{15}-16\sqrt{6}}{256}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
C &= \gamma_-(1)\gamma_+(1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},2,1,1,1)\\
&\quad\times \frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_+(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\\
&\quad\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_+(3)
\end{aligned}
$$

$$\times \frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},0,1,1,1)\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},4,1,1,1)\mathrm{Tet}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\frac{\mathrm{Tet}\begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_1\theta(3,1,2)}$$

$$= 0 - \frac{1}{2}\frac{\frac{\sqrt{3}}{3}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\frac{1}{2}}\frac{\frac{\sqrt{-6}}{6}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{5}{2}\sqrt{3}}}{\sqrt{-2}/3}\frac{3\frac{3}{2}}{(-2)3} + 0$$

$$= \frac{3\sqrt{15}}{256}(\hbar k)^3,$$

$$D = \gamma_-(1)\gamma_-(1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)\frac{N_{10}V_{10}^{10}}{N_{10}}(\hat{1},0,1,1,1)$$

$$\times \frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_-(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)$$

$$\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_1\theta(1,1,2)} + \gamma_-(1)\gamma_-(3)$$

$$\times \frac{N_{32}V_{10}^{32}}{N_{10}}(\hat{1},0,1,1,1)\frac{N_{10}V_{32}^{10}}{N_{32}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}}{\Delta_1\theta(3,1,2)}$$

$$= 0 + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{\frac{\sqrt{3}}{3}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\frac{1}{2}}\frac{\frac{1}{2}\frac{(\hbar k)^{3/2}}{4}\sqrt{\frac{\sqrt{3}}{2}}}{\sqrt{3}/3}\frac{3\cdot 3}{(-2)\cdot 3} + 0$$

$$= -\frac{3\sqrt{3}}{256}(\hbar k)^3.$$

Taking the values of A, B, C, D into (56), the expectation value for "00" metric component is

$$M(S_0,S_0) = -\frac{8}{(\hbar k)^2}\left[\frac{-15\sqrt{3}+3\sqrt{15}-16\sqrt{6}+3\sqrt{15}-3\sqrt{3}}{256}\right]\hbar k$$
$$= 1.473\hbar k. \tag{57}$$

For the other metric matrix diagonal components similar calculations load to the results:

$$M(s_1,s_1) = 0.250\hbar k, \qquad M(s_2,s_2) = 1.473\hbar k, \qquad M(s_3,s_3) = 0.250\hbar k.$$

9 Expectation Values of Metric Matrix Off-Diagonal Component Operators

As an example, we first give the computation of expectation value for metric component "01". From (16), the expression of this metric component is

$$M(s_0,s_1) = \frac{8}{(\hbar k)^2}\sum_{q=\pm 1}\sum_{g=\pm 1}\sum_{t}\gamma_q(p)\gamma_g(p)\left(\frac{N_k}{N_m}\right)(p)\left\{\frac{N_{pt}V_{pk}^{pt}}{N_{pk}}(\hat{1},p+q,p,p,p)\right.$$

$$
\begin{aligned}
&\times\left(\sum_{\gamma}\begin{Bmatrix} p & p & r \\ p & p & t \end{Bmatrix}\left(\frac{N_{pm}V_{pr}^{pm}}{N_{pr}}\right)(\hat{1}, p+g, p, p, p)\right) \\
&+\left(\sum_{l}\begin{Bmatrix} p & p & l \\ p & p & k \end{Bmatrix}\left(\frac{N_{pt}V_{pl}^{pt}}{N_{pl}}\right)(\hat{1}, p+q, p, p, p)\right) \\
&\times\left(\sum_{l''}\begin{Bmatrix} s & p & l'' \\ p & p & t \end{Bmatrix}\sum_{l'}\begin{Bmatrix} p & p & l' \\ p & s & l'' \end{Bmatrix}\sum_{l}\begin{Bmatrix} p & p & l \\ s & p & l' \end{Bmatrix}\right. \\
&\times\left.\left(\frac{N_{pm}V_{pl}^{pm}}{N_{pl}}\right)(\hat{1}, p+g, p, p, p)\right)\Bigg\} \\
&\times\frac{\mathrm{Tet}\begin{bmatrix} 2 & p & p+q \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} p & p & p+q \\ 1 & 1 & 2 \end{bmatrix}}{\theta(2,2,p)\theta(p,2,p)}.
\end{aligned} \tag{58}
$$

Let the two terms in (58) be M_{10} and M_{01} respectively. Then for M_{10}, expending it with $\sum_{q=\pm 1}$ and $\sum_{g=\pm 1}$, we have

$$
\begin{aligned}
M_{10} = \frac{8}{(\hbar k)^2}\sum_{t}\Bigg[&\gamma_+(1)\gamma_+(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(\hat{1},2,1,1,1)\left(\sum_{r}\begin{Bmatrix} 1 & 1 & r \\ 1 & 1 & t \end{Bmatrix}\frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},2,1,1,1)\right) \\
&\times\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} + \gamma_+(1)\gamma_-(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(\hat{1},2,1,1,1) \\
&\times\left(\sum_{r}\begin{Bmatrix} 1 & 1 & r \\ 1 & 1 & t \end{Bmatrix}\frac{N_{10}V_{pr}^{1t}}{N_{1r}}(\hat{1},0,1,1,1)\right)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} \\
&+\gamma_-(1)\gamma_+(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(\hat{1},0,1,1,1)\left(\sum_{r}\begin{Bmatrix} 1 & 1 & r \\ 1 & 1 & t \end{Bmatrix}\frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},2,1,1,1)\right) \\
&\times\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} + \gamma_-(1)\gamma_-(1)\frac{N_{1t}V_{10}^{1t}}{N_{10}}(1,0,1,1,1) \\
&\times\left(\sum_{r}\begin{Bmatrix} 1 & 1 & r \\ 1 & 1 & t \end{Bmatrix}\frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},0,1,1,1)\right)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\Bigg].
\end{aligned} \tag{59}
$$

Denote the four terms in the brackets of (59) as E, F, G, H respectively, because the color t may only be taken as 0 or 2 in this case, and for every value of t the color r may be 0 or 2, after some calculation, the following results are got:

$$
\begin{aligned}
E &= \gamma_+(1)\gamma_+(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\begin{Bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{Bmatrix}\\
&\quad\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\
&= \frac{5\sqrt{3}}{256}(\hbar k)^3,\\
F &= \gamma_+(1)\gamma_-(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\begin{Bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{Bmatrix}\\
&\quad\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\
&= -\frac{2\sqrt{15}}{512}(\hbar k)^3,\\
G &= \gamma_-(1)\gamma_+(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\begin{Bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{Bmatrix}\\
&\quad\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\
&= -\frac{2\sqrt{15}}{512}(\hbar k)^3,\\
H &= \gamma_-(1)\gamma_-(1)\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)\begin{Bmatrix} 1 & 1 & 2\\ 1 & 1 & 2\end{Bmatrix}\\
&\quad\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\
&= \frac{5\sqrt{3}}{256}(\hbar k)^3.
\end{aligned}
$$

Putting E, F, G, H into (59), we have $M_{10} \doteq 0.083\hbar k$.

For M_{01}, out of same argument, we have

$$
\begin{aligned}
M_{01} = \frac{8}{(\hbar k)^2}\sum_{g=\pm 1}\sum_{t}\Bigg[&\gamma_+(1)\gamma_+(1)\Bigg(\sum_{l}\begin{Bmatrix} 1 & 1 & l\\ 1 & 1 & 0\end{Bmatrix}\Bigg(\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},2,1,1,1)\Bigg)\Bigg)\\
&\times\Bigg(\sum_{r''}\begin{Bmatrix} 1 & 1 & t''\\ 1 & 1 & t\end{Bmatrix}\sum_{r'}\begin{Bmatrix} 1 & 1 & r'\\ 1 & 1 & r''\end{Bmatrix}\sum_{r}\begin{Bmatrix} 1 & 1 & r\\ 1 & 1 & r'\end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},2,1,1,1)\Bigg)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2'\end{bmatrix}}{\theta(1,2,1)\theta(1,1,2)}\\
&+\gamma_+(1)\gamma_-(1)\Bigg(\sum_l\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\Bigg(\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},2,1,1,1)\Bigg)\Bigg)\\
&\times\Bigg(\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_r\begin{Bmatrix}1&1&r\\1&1&r'\end{Bmatrix}\\
&\times \frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},0,1,1,1)\Bigg)\frac{\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,2,1)}\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)}\\
&+\gamma_-(1)\gamma_+(1)\Bigg(\sum_l\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},0,1,1,1)\Bigg)\\
&\times\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_r\begin{Bmatrix}1&1&r\\1&1&r'\end{Bmatrix}\\
&\times \frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}s&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\\
&+\gamma_-(1)\gamma_-(1)\Bigg(\sum_l\begin{Bmatrix}1&1&l\\1&1&0\end{Bmatrix}\frac{N_{1t}V_{1l}^{1t}}{N_{1l}}(\hat{1},0,1,1,1)\Bigg)\\
&\times\Bigg(\sum_{r''}\begin{Bmatrix}1&1&r''\\1&1&t\end{Bmatrix}\sum_{r'}\begin{Bmatrix}1&1&r'\\1&1&r''\end{Bmatrix}\sum_r\begin{Bmatrix}1&1&r\\0&1&r'\end{Bmatrix}\\
&\times \frac{N_{10}V_{pr}^{10}}{N_{pr}}(\hat{1},0,1,1,1)\Bigg)\frac{\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1&1&0\\1&1&2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}\Bigg]. \qquad (60)
\end{aligned}
$$

Let the four terms in the brackets of above expression be E', F', G', H', after similar calculation as M_{10}, the results are

$$
\begin{aligned}
E'=&\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\Bigg(\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\\
&\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)+\begin{Bmatrix}1&1&0\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\\
&\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)+\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&0\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\\
&\times\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)+\begin{Bmatrix}1&1&2\\1&1&0\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\Bigg)\frac{\frac{3}{2}\cdot\frac{3}{2}}{3\cdot 3} + \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1) \\
&\times \left(\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} + \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\right. \\
&\left. + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\right) \\
&\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} \\
&= \frac{10\sqrt{3}}{512}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
F' &= \gamma_+(1)\gamma_-(1)\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1) \\
&\times \left(\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} + \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\right. \\
&\left. + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\right) \\
&\times \frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} + \gamma_+(1)\gamma_-(1)\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix} \\
&\times \frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},2,1,1,1)\left(\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\right. \\
&+ \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix} + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} \\
&\left. + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}\right)\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1) \\
&\times \frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} \\
&= -\frac{\sqrt{15}}{256}(\hbar k)^3,
\end{aligned}
$$

$$
\begin{aligned}
G' &= \gamma_-(1)\gamma_+(1)\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1) \\
&\times \frac{\mathrm{Tet}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\mathrm{Tet}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)} + \gamma_-(1)\gamma_+(1)\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix}\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)
\end{aligned}
$$

$$\times \frac{4}{8}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},2,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}$$

$$= -\frac{\sqrt{15}}{256}(\hbar k)^3,$$

$$H' = \gamma_-(1)\gamma_-(1)\begin{Bmatrix}1 & 1 & 2\\ 1 & 1 & 0\end{Bmatrix}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)$$

$$\times \frac{\mathrm{Tet}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)}\frac{\mathrm{Tet}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)} + \gamma_-(1)\gamma_-(1)\begin{Bmatrix}1 & 1 & 0\\ 1 & 1 & 0\end{Bmatrix}\frac{N_{12}V_{10}^{12}}{N_{10}}(\hat{1},0,1,1,1)$$

$$\times \frac{4}{8}\frac{N_{10}V_{12}^{10}}{N_{12}}(\hat{1},0,1,1,1)\frac{\mathrm{Tet}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\mathrm{Tet}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}}{\theta(1,1,2)\theta(1,1,2)}$$

$$= \frac{\sqrt{3}}{256}(\hbar k)^3.$$

Sum up the values of E', F', G' and H', we obtain the value of M_{01} as $M_{01} \doteq 0.083\hbar k$.

Combine the values of M_{10} and M_{01}, the final result is

$$M(s_0, s_1) = M_{10} + M_{01} = 0.165\hbar k. \tag{61}$$

For the expectation values of other metric matrix off-diagonal components we directly give them as follows:

$$M(s_0,s_2) = 0.330\hbar k, \qquad M(s_0,s_3) = 0.165\hbar k, \qquad M(s_1,s_2) = 0.165\hbar k,$$
$$M(s_1,s_3) = -0.165\hbar k, \qquad M(s_2,s_3) = 0.165\hbar k.$$

10 Spin-Geometry of Gaussian Weave State

Employing the obtained expectation value of metric operator with respect to the vertex ϕ_k carried on basis state ψ_p framing the Gauss weave state at vertex υ, we now approach the spin-geometry of the weave state. According to the Penrose spin theorem [6], the angles between tangent directions of the phase configurations of edges adjacent to the vertex υ are defined as

$$\theta(s_\alpha, s_\beta) = \cos^{-1}(M(s_\alpha, s_\beta)/\sqrt{M(s_\alpha, s_\alpha)}\sqrt{M(s_\beta, s_\beta)}). \tag{62}$$

10.1 Case of $k = 1$

Collecting the expectation values of metric component operators above obtained in the case of $k = 0$, the metric matrix is

$$M(s_\alpha, s_\beta) = \hbar k\begin{bmatrix}1.473 & 0.165 & 0.330 & 0.165\\ 0.165 & 0.250 & 0.165 & -0.165\\ 0.330 & 0.165 & 1.473 & 0.165\\ 0.165 & -0.165 & 0.165 & 0.250\end{bmatrix}. \tag{63}$$

Introducing the corresponding values from (63) into (62), the angles are

$$\theta(s_0,s_1)=\cos^{-1}0.274=74.12^\circ,\qquad \theta(s_0,s_2)=\cos^{-1}0.224=77.05^\circ,$$
$$\theta(s_0,s_3)=\cos^{-1}0.274=74.12^\circ,\qquad \theta(s_1,s_2)=\cos^{-1}0.274=74.12^\circ,$$
$$\theta(s_1,s_3)=\cos^{-1}(-0.656)=130.99^\circ,\qquad \theta(s_2,s_3)=\cos^{-1}0.274=74.12^\circ.$$

The values of length of tangent vectors at υ can be computed as

$$|\dot{s}_0|=1.214(\hbar k)^{1/2},\qquad |\dot{s}_1|=0.5(\hbar k)^{1/2},\qquad |\dot{s}_2|=1.214(\hbar k)^{1/2},\qquad |\dot{s}_3|=0.5(\hbar k)^{1/2}$$

and an average length of the tangent vectors may be used to assign a distance between vertices:

$$\frac{1}{4}\sum_\alpha\sqrt{M(s_\alpha,s_\alpha)}=0.875(\hbar k)^{1/2}.$$

For Gaussian weave state W, in order to avoid contribution from higher color of edge, Gaussian width is usually taken as $\lambda=3/4$. In this case, the coefficients C_p of higher color more than $p=3$ are negligible, and a peak is centered at the fundamental representation ($p=1$) of $su(2)$.

10.2 Case of $k=2$

Continue the above calculation for the case of $k=2$, the following results are obtained:

the metric matrix of expectation value of operator $\hat{M}(s_\alpha,s_\beta)$ is

$$M(s_\alpha,s_\beta)=\hbar k\begin{bmatrix}1.309 & 0 & 0.330 & 0\\ 0 & 1.642 & 0 & 0.165\\ 0.330 & 0 & 1.309 & 0\\ 0 & 0.165 & 0 & 1.642\end{bmatrix},$$

the angles between the tangent direction of phase configurations of edges are

$$\theta(s_0,s_1)=\cos^{-1}0=90^\circ,\qquad \theta(s_0,s_2)=\cos^{-1}0.165=80.5^\circ,$$
$$\theta(s_0,s_3)=\cos^{-1}0=90^\circ,\qquad \theta(s_1,s_2)=\cos^{-1}0=90^\circ,$$
$$\theta(s_1,s_3)=\cos^{-1}0.101=84.2^\circ,\qquad \theta(s_2,s_3)=\cos^{-1}0=90^\circ,$$

the values of length of tangent vectors at vertex υ are

$$|\dot{S}_0|=1.144(\hbar k)^{1/2},\qquad |\dot{S}_1|=1.281(\hbar k)^{1/2},$$
$$|\dot{S}_2|=1.144(\hbar k)^{1/2},\qquad |\dot{S}_3|=1.281(\hbar k)^{1/2}$$

and the average length of the tangent vectors is

$$\frac{1}{4}\sum_\alpha\sqrt{M(s_\alpha,s_\alpha)}=1.213(\hbar k)^{1/2}.$$

The results above calculated about the metric matrix, angles between tangent directions, and the lengths of tangent vectors are all symmetric under the index change $0\leftrightarrow 2$, $1\leftrightarrow 3$.

It is found that the diagonal elements of metric matrix are surely positive definite, and the lengths of the tangent vectors at υ are positive. The expectation value of the metric operator with respect to the Gauss weave state may well determine the scale of the state. This issue relevant to spin-geometry may be used to investigate the weaving of space, which describes a semi-classical picture in loop gravity, further.

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